Understanding sparsity properties of frames using decomposition spaces

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Outline

1. Different representation systems and their applications (5 slides)
2. Structured systems & Decomposition spaces (2 slides)
3. Sparsity properties characterized by decomposition spaces (7 slides)

Outline

The Fourier basis: The mother of all representation systems

Consider Fourier basis $(e_n)_{n\in\mathbb{Z}^d} := \left(e^{2\pi i\langle n,\bullet\rangle}\right)_{n\in\mathbb{Z}^d}$ on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Nice properties:

- Fourier basis is an **orthonormal basis** for $L^2(\mathbb{T}^d)$.
- Smooth functions are **sparse** in the Fourier basis.

  For example: If $g \in C^k\left(\mathbb{T}^d\right)$, then $(\widehat{g}(n))_{n\in\mathbb{Z}^d} \in \ell^p\left(\mathbb{Z}^d\right)$ for $p > \frac{2d}{d+2k}$.
- Fourier basis **diagonalizes** translation invariant operators.

But:

- The coefficients $\widehat{g}(n) = \langle g, e_n\rangle$ are **non-local**.

  $\rightsquigarrow$ If g is discontinuous **at a single point**, then $(\widehat{g}(n))_{n\in\mathbb{Z}^d} \notin \ell^1\left(\mathbb{Z}^d\right)$.
- For the Fourier transform on $\mathbb{R}^d$:
  - The index set of $(e^{2\pi i\langle \xi,\bullet\rangle})_{\xi\in\mathbb{R}^d}$ is **uncountable**.
  - We have $e^{2\pi i\langle \xi,\bullet\rangle} \notin L^p\left(\mathbb{R}^d\right)$ for $\xi \in \mathbb{R}^d$ and $p < \infty$.
  - The coefficients $\widehat{g}(\xi) = \langle g, e^{2\pi i\langle \xi,\bullet\rangle}\rangle$ depend **non-locally** on g.

A wish list for representation systems

- $\Phi = (\varphi_i)_{i\in I}$ should be a **frame** for Hilbert space $\mathcal{H}$, i.e.:
  1. $f \in \mathcal{H}$ can be stably recovered from the **analysis coefficients** $(\langle f, \varphi_i \rangle)_{i\in I}$.
  2. The **synthesis map**

     $$S_\Phi : \ell^2(I) \to \mathcal{H}, \quad (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \varphi_i$$

     is bounded and surjective.

  Note: Each of these prop. equivalent to $\|f\|_{\mathcal{H}}^2 \asymp \sum_{i\in I} |\langle f, \varphi_i \rangle|^2$ for all $f \in \mathcal{H}$.

- Φ should yield **sparse representations for a function class** $\mathcal{C}$ of interest.
  1. **Analysis sparsity** means $(\langle f, \varphi_i \rangle)_{i\in I} \in \ell^p(I)$.
  2. **Synthesis sparsity** means $f = \sum_{i\in I} c_i \varphi_i$ with $(c_i)_{i\in I} \in \ell^p(I)$.

- Φ should be **"compatible" with a class of operators** of interest (e.g.: operators in the class preserve sparsity w.r.t. Φ).

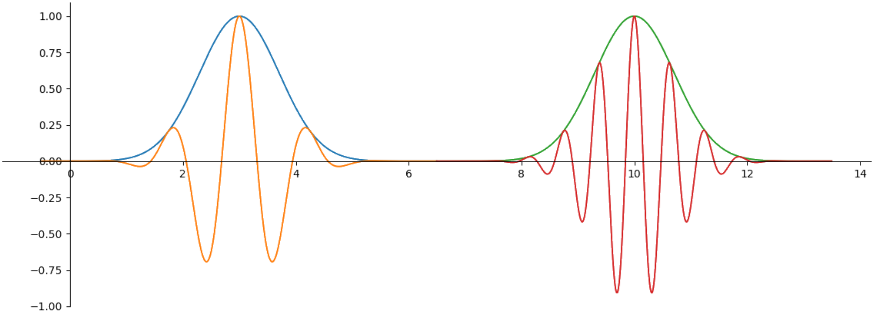
- **Characterization of function spaces** using the coefficients $(\langle f, \varphi_i \rangle)_{i\in I}$.
- Φ should be **efficiently implementable**.
- Φ should be useful **feature extractor** (e.g. for edge detection).

Example 1: Gabor frames

Given a **prototype** $\gamma \in L^2(\mathbb{R}^d)$ and **sampling densities** $a, b > 0$, the associated **Gabor system** is

$$\mathcal{G}(\gamma; a, b) := \left\{ \gamma^{[n,k;a,b]} := L_{bk}\left[M_{an}\,\gamma\right] = e^{2\pi i \langle an, \bullet - bk \rangle} \cdot \gamma(\bullet - bk) \,:\, n, k \in \mathbb{Z}^d \right\}.$$

Illustration:



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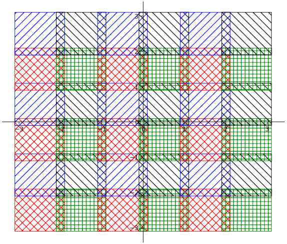
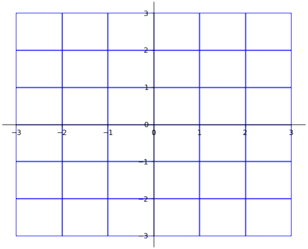
Applications:

- Feature extraction: Gabor transform $\approx$ score sheet 
- Gabor frames characterize **modulation spaces**, used to study ΨDOs.

Frequency concentration: If $\operatorname{supp} \widehat{\gamma} \subset Q$, then $\operatorname{supp} \mathcal{F}\, \gamma^{[n,k;a,b]} \subset Q + an$.



or



Given **low-pass filter** $\varphi \in L^2(\mathbb{R}^d)$, **mother wavelet** $\gamma \in L^2(\mathbb{R}^d)$, and sampling density $\delta > 0$, the associated **wavelet system** is

$$\begin{aligned}\mathcal{W}(\varphi,\gamma;\delta) := & \left\{\gamma^{[-1,k;\delta]} := L_{\delta k}\varphi \,:\, k \in \mathbb{Z}^d\right\} \\ & \cup \left\{\gamma^{[j,k;\delta]} := 2^{jd/2}\cdot\gamma(2^j \bullet - \delta k) \,:\, j \in \mathbb{N}_0, k \in \mathbb{Z}^d\right\}.\end{aligned}$$

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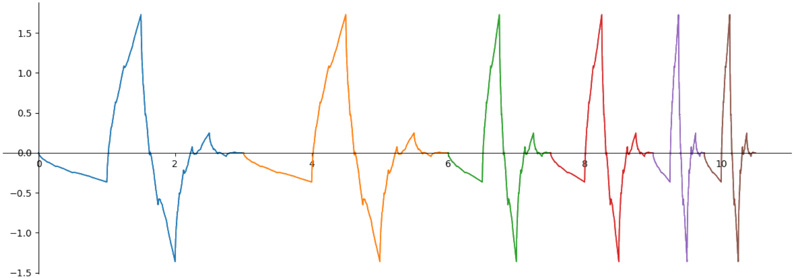

Example 2: Wavelet frames

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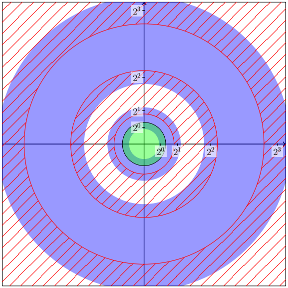
Applications:

- Feature extraction: Wavelet transform can detect singularities of functions.
- Sparsity: Piecewise smooth functions have sparse wavelet transform.
- Wavelet frames characterize **Besov spaces** and **Triebel-Lizorkin spaces**, used to study Calderon-Zygmund operators.

Frequency concentration:

If $\operatorname{supp}\widehat{\gamma} \subset Q$, then $\operatorname{supp}\mathcal{F}\gamma^{[j,k;\delta]} \subset 2^j Q$ for $j \in \mathbb{N}_0$.



Example 3: Shearlet frames

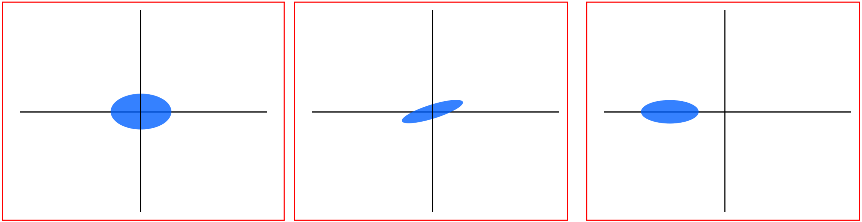
Given **low-pass filter** φ, **mother shearlet** γ, and sampling density $\delta > 0$, the associated **cone-adapted shearlet system** is

$$\mathrm{SH}(\varphi,\gamma;\delta) := \left\{ \gamma^{[-1,k;\delta]} := L_{\delta k}\varphi \,:\, k \in \mathbb{Z}^2 \right\} \cup \left\{ \gamma^{[i,k;\delta]} := |\det T_i|^{1/2} \cdot \gamma(T_i^T \bullet - \delta k) \,:\, i \in I_0,\, k \in \mathbb{Z}^2 \right\}$$

with $I_0 := \left\{ (j,\ell,\iota) \in \mathbb{N}_0 \times \mathbb{Z} \times \{0,1\} \,:\, |\ell| \leq \lceil 2^{j/2} \rceil \right\}$ and

$$T_{j,\ell,\iota} = R^\iota \cdot P_j \cdot S_\ell \quad \text{with} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \quad S_\ell = \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}.$$

Illustration (images courtesy of Philipp Petersen)



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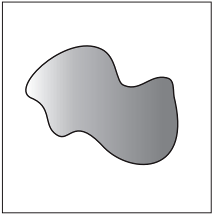
Applications:

- Feature extraction: Decay of shearlet transform characterizes wavefront set.
- Sparsity: The shearlet coefficients of **cartoon-like functions** are sparse.



- What kind of spaces are characterised by shearlets? What kinds of operators can be understood using shearlets?

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Applications:

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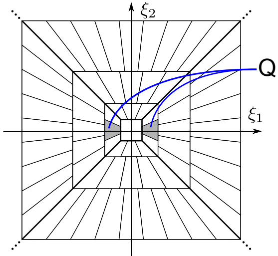


Outline

Structured systems

A **structured system** is of the form $\Gamma^{(\delta)} = \left(\gamma^{[i,k;\delta]}\right)_{i\in I, k\in\mathbb{Z}^d}$ with

$$\boxed{\gamma^{[i,k;\delta]} = |\det T_i|^{\frac{1}{2}} \cdot L_{\delta\cdot T_i^{-T}\cdot k}\left[M_{b_i}\left(\gamma \circ T_i^T\right)\right]}$$

for some prototype γ and a family of affine maps $(T_i \bullet + b_i)_{i\in I}$.

Note: All of the considered systems are (more of less) of this form!

Frequency concentration of structured system: $\gamma^{[i]} := \gamma^{[i,0;\delta]}$ satisfies

$$\mathcal{F}\gamma^{[i]} = |\det T_i|^{-1/2} \cdot L_{b_i}\left(\widehat{\gamma} \circ T_i^{-1}\right).$$

If $\widehat{\gamma}$ is concentrated in $Q \subset \mathbb{R}^d$, then $\mathcal{F}\gamma^{[i]}$ is concentrated in $Q_i := T_i Q + b_i$.

$\rightsquigarrow \mathcal{Q} = (Q_i)_{i\in I}$ is the **structured covering** associated to the structured system.

Goal: Find family of Banach spaces $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, parametrized by p, q, w, so that

$$f \in \mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) \quad \Longleftrightarrow \quad f \text{ sparse with respect to } \Gamma^{(\delta)}.$$

Decomposition spaces

Consider **structured admissible covering** $\mathcal{Q} = (Q_i)_{i\in I}$ of $\mathbb{R}^d$. This means:

- There is a fixed open, bounded **base set** $Q \subset \mathbb{R}^d$ with

$$Q_i = T_i Q + b_i \qquad \forall i \in I,$$

- $|i^*| \leq N$ for all $i \in I$, with $i^* := \{\ell \in I \,:\, Q_\ell \cap Q_i \neq \varnothing\}$,
- additional technical conditions (almost always satisfied in practice).

Choose a **regular partition of unity** $\Phi = (\varphi_i)_{i\in I}$ subordinate to $\mathcal{Q}$.

$\varphi_i \in C_c^\infty(Q_i)$, $\sum_{i\in I} \varphi_i \equiv 1$ and some other technical condition.

Choose a **$\mathcal{Q}$-moderate weight** $w = (w_i)_{i\in I}$, i.e., $w_i \leq C \cdot w_j$ if $Q_i \cap Q_j \neq \varnothing$.

For $p, q \in (0,\infty]$, and $g \in \mathcal{R}$, define the **decomposition space (quasi)-norm**

$$\|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_w^q)} := \left\| \left(w_i \cdot \left\| \mathcal{F}^{-1}(\varphi_i \cdot \widehat{g}) \right\|_{L^p} \right)_{i\in I} \right\|_{\ell^q} \in [0,\infty].$$

The **decomposition space** determined by $\mathcal{Q}, p, q, w$ is

$$\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q) := \left\{ g \in \mathcal{R} \,:\, \|g\|_{\mathcal{D}(\mathcal{Q},L^p,\ell_w^q)} < \infty \right\}.$$

Here $\mathcal{R}$ is a suitable reservoir of distributions (think: $\mathcal{R} = \mathcal{S}'(\mathbb{R}^d)$).

Outline

Banach frames and atomic decompositions

$(\varphi_i)_{i\in I} \subset X'$ is a **Banach frame** for the B-space X, with coeff. space $Y_d \leq \mathbb{C}^I$ if

1. Y_d is solid.
2. The **analysis operator**

$$A : X \to Y_d, x \mapsto (\langle x, \varphi_i \rangle)_{i\in I}$$

   is well-defined and bounded.
3. There is a bounded **reconstruction operator** $R : Y_d \to X$ with $R \circ A = \mathrm{id}_X$.

$(\varphi_i)_{i\in I} \subset X$ is an **atomic decomposition** for X, with coeff. space $Y_d \leq \mathbb{C}^I$ if

1. Y_d is solid.
2. The **synthesis operator**

$$S : Y_d \to X, (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \varphi_i$$

   is well-defined and bounded (convergence in suitable topology).
3. There is a bounded **coefficient operator** $C : X \to Y_d$ with $S \circ C = \mathrm{id}_X$.

Hope: If γ is nice, then $\Gamma^{(\delta)} = \left(\gamma^{[i,k;\delta]}\right)_{i\in I, k\in\mathbb{Z}^d}$ is a BFD for $\mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$, where

$$\gamma^{[i,k;\delta]} = |\det T_i|^{\frac{1}{2}} \cdot L_{\delta \cdot T_i^{-T} k} \left[M_{b_i}\left(\gamma \circ T_i^T\right)\right].$$

Question: What is the associated sequence space $C_w^{p,q} \leq \mathbb{C}^{I\times\mathbb{Z}^d}$?

Answer: $C_w^{p,q} = \left\{c \in \mathbb{C}^{I\times\mathbb{Z}^d} : \|c\|_{C_w^{p,q}} < \infty\right\}$, where

$$\left\|(c_k^{(i)})_{i\in I, k\in\mathbb{Z}^d}\right\|_{C_w^{p,q}} = \left\|\left(|\det T_i|^{\frac{1}{2}-\frac{1}{p}} \cdot w_i \cdot \left\|(c_k^{(i)})_{k\in\mathbb{Z}^d}\right\|_{\ell^p}\right)_{i\in I}\right\|_{\ell^q}.$$

Question: In what sense does γ have to be "nice"?

Note: Even for characterizing L^2, the required "niceness" depends heavily on $\mathcal{Q}$:

- For Gabor systems: Sufficient if γ belongs to the Wiener space.
- For wavelets, γ has to have vanishing moments.

Structured Banach frames — The theorem

Theorem (FV; 2016)

Let $w = (w_i)_{i\in I}$ be $\mathcal{Q}$-moderate and let $p, q \in (0,\infty]$.
There are $N \in \mathbb{N}$ and $\sigma, \tau > 0$ (only dep. on p, q, d) with the following property:

If
- $\gamma \in C_c^1(\mathbb{R}^d)$,
- $\widehat{\gamma}(\xi) \neq 0$ *for all* $\xi \in \overline{Q}$,
- *and if*

$$\sup_{i\in I} \sum_{j\in I} M_{j,i} < \infty \quad \text{and} \quad \sup_{j\in I} \sum_{i\in I} M_{j,i} < \infty,$$

with

$$M_{j,i} := \left(\frac{w_j}{w_i}\right)^{\tau} \cdot (1 + \| T_j^{-1} T_i \|)^{\sigma} \cdot \left(\fint_{Q_i} \max_{\substack{|\alpha|\leq N \\ |\beta|\leq 1}} \left| [\partial^{\alpha} \widehat{\partial^{\beta} \gamma}] \left(T_j^{-1} (\xi - b_j) \right) \right| \mathrm{d}\xi \right)^{\tau},$$

then there is $\delta_0 > 0$, such that:

For $0 < \delta \leq \delta_0$, the family $\left(L_{\delta \cdot T_i^{-T} k}\, \widetilde{\gamma^{[i]}} \right)_{i,k}$ is Banach frame for $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$

with coefficient space $C_w^{p,q}$. *Here:* $\tilde{g}(x) = g(-x)$.

Theorem (FV; 2016)

Under similar conditions on γ as before, there is $\delta_0 > 0$ such that:

$$\text{For } 0 < \delta \leq \delta_0, \text{ the family } \Gamma^{(\delta)} = \left(\gamma^{[i,k;\delta]}\right)_{i,k} \text{ is an a.d. of } \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right)$$

with coefficient space $C_w^{p,q}$.

Observation: For $w_i = |\det T_i|^{\frac{1}{p}-\frac{1}{2}}$, we have $C_w^{p,p} = \ell^p\left(I \times \mathbb{Z}^d\right)$.

Thus: For $0 < p \leq 2$, for sufficiently nice γ, and $\delta > 0$ small, we have

$$\boxed{\begin{aligned} \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^p\right) &= \left\{ f \in L^2\left(\mathbb{R}^d\right) : \left(\left\langle f, \gamma^{[i,k;\delta,*]}\right\rangle\right)_{i \in I, k \in \mathbb{Z}^d} \in \ell^p\left(I \times \mathbb{Z}^d\right) \right\} \\ &= \left\{ \sum_{i,k} c_k^{(i)} \cdot \gamma^{[i,k;\delta]} : \left(c_k^{(i)}\right)_{i \in I, k \in \mathbb{Z}^d} \in \ell^p\left(I \times \mathbb{Z}^d\right) \right\}, \end{aligned}}$$

with $\mathcal{Q} = \left(T_i Q + b_i\right)_{i \in I}$ and $\gamma^{[i]} = |\det T_i|^{\frac{1}{2}} \cdot M_{b_i}\left[\gamma \circ T_i^T\right]$, as well as

$$\gamma^{[i,k;\delta]} = L_{\delta \cdot T_i^{-T} k}\, \gamma^{[i]} \quad \text{and} \quad \gamma^{[i,k;\delta,*]} = L_{\delta \cdot T_i^{-T} k}\, \widetilde{\gamma^{[i]}} .$$

New proofs of classical results

Corollary

For ***Besov spaces*** $B_s^{p,q}(\mathbb{R}^d)$*:*

- *Structured system* $\rightsquigarrow$ ***Wavelet system****.*
- *"Horrible condition"* $\rightsquigarrow$ *smoothness + localization +* ***vanishing moments****.*
- *Precisely: Suffices to have*

$$|\partial^\alpha \widehat{\gamma}(\xi)| \lesssim (1+|\xi|)^{-L_1} \cdot \min\left\{1, |\xi|^{L_2}\right\} \qquad \text{for } |\alpha| \leq \left\lceil \frac{d+\varepsilon}{\min\{1,p\}} \right\rceil,$$

  with $L_2 \geq 0$ *and*

  - $L_2 > s$ *to get Banach frames,*
  - $L_2 > \left(p^{-1} - 1\right)_+ \cdot d - s$ *to get atomic decompositions.*

Corollary

For ***modulation spaces*** $M_s^{p,q}(\mathbb{R}^d)$*:*

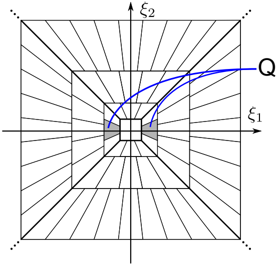
- *Structured system:* ***Gabor system****.*
- *"Horrible condition"* $\rightsquigarrow$ *smoothness + localization.*

Shearlet smoothness spaces

This is joint work with **Anne Pein**

Shearlet smoothness spaces

The **shearlet covering** $\mathcal{S} = (T_i Q + b_i)_{i \in I}$:



The spaces $\mathscr{S}_s^{p,q}(\mathbb{R}^2) = \mathcal{D}(\mathcal{S}, L^p, \ell_{w^s}^q)$ are called **shearlet smoothness spaces** **(Labate, Mantovani, Negi; 2013).**

The weight w is chosen such that $w_i \sim 1 + |\xi|$ for $\xi \in Q_i$.

Observation: Structured family generated by ψ is a **cone-adapted shearlet system** **(Guo, Labate, Kutyniok; 2006)**:

$$\Psi^{(\delta)} := \left(\psi^{[i,k;\delta]}\right)_{i \in I, k \in \mathbb{Z}^2} = \mathrm{SH}\left(M_{b_0}(\psi \circ T_0^T), \psi; \delta\right).$$

Theorem (Pein, FV; 2017)

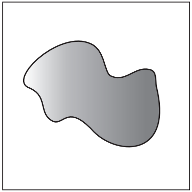
Let $p_0, q_0 \in (0,1]$ and $s_0 \geq 0$. There are $N_1, N_2 \in \mathbb{N}$ such that if $\psi_1, \psi_2 \in C_c^{N_1}(\mathbb{R})$ and $\psi = \psi_1 \otimes \psi_2$ with

$$\widehat{\psi}(\xi) \neq 0 \text{ for } \xi \in \overline{Q} \quad \text{and} \quad \boxed{\left.\frac{\mathrm{d}^\ell}{\mathrm{d}\,\xi^\ell}\right|_{\xi=0} \widehat{\psi_1}(\xi) = 0 \text{ for } \ell = 0, \ldots, N_2}$$

*then there is $\delta_0 > 0$, such that $\Psi^{(\delta)}$ is a Banach frame **and** an atomic decomposition for $\mathscr{S}_s^{p,q}(\mathbb{R}^2)$ for all $p \geq p_0$, $q \geq q_0$, $|s| \leq s_0$, and $0 < \delta \leq \delta_0$.*

Application: Approximation of cartoon-like functions

We consider C^2 **cartoon-like functions** $f \in \mathcal{E}^2$:



(image courtesy of Gitta Kutyniok)

Previously known (Guo, Labate, Lim, Kutyniok et al.): If ψ is a nice mother shearlet, then

- the **analysis coefficients** of f w.r.t. $\mathrm{SH}(\psi;\delta)$ are **ℓ^p-summable** for $p > \frac{2}{3}$.
- For suitable linear combination f_N of N elements of the **dual shearlet frame**:

$$\|f - f_N\|_{L^2} \leq C_{\delta,\psi} \cdot N^{-1} \cdot (1 + \ln N)^{3/2}.$$

New result (Pein, FV; 2017)

For suitable linear comb. g_N of N elements **of the shearlet frame** $\mathrm{SH}(\psi;\delta)$:

$$\|f - g_N\|_{L^2} \leq C_{\varepsilon,\delta,\psi} \cdot N^{-(1-\varepsilon)} \qquad \forall \varepsilon \in (0,1) \text{ and } N \in \mathbb{N}.$$

Conclusion

General framework:

- Frame construction $\longleftrightarrow$ Frequency covering $\longleftrightarrow$ Decomposition space
- Membership $f \in \mathcal{D}\left(\mathcal{Q}, L^p, \ell_w^q\right) \Longleftrightarrow$ Sparsity of f w.r.t. given frame.

Special cases:

- We recover known results about wavelet and Gabor frames.
- Novel results for cone-adapted shearlets.

Story for another day:

Embedding theory for decomposition spaces.

$\rightsquigarrow$ (Non)-transference of sparsity from one system to another.

Future work:

- Study **operators** on decomposition spaces.
- Try to find **"intrinsic" characterizations** of decomposition spaces (e.g.: characterization of Besov spaces using moduli of continuity).
- Extend framework to the case of **Triebel-Lizorkin type spaces**.
- Study decomposition spaces on **(bounded) domains**.

Thank you!

Questions, comments, counterexamples?

☺

Atomic decompositions: "Similar conditions"?!

Theorem (FV; 2016)

Let $w = (w_i)_{i \in I}$ *be* $\mathcal{Q}$*-moderate, and let* $p, q \in (0, \infty]$.

There are $N \in \mathbb{N}$ *and* $\sigma, \tau > 0$ *and* $\vartheta \geq 0$ *(only dep. on* p, q, d*) with the following:*

Assume

- $\gamma = \gamma^{(1)} * \gamma^{(2)}$ *with* $\gamma^{(1)} \in C_c(\mathbb{R}^d)$ *and* $\gamma^{(2)} \in C_c^1(\mathbb{R}^d)$,
- $\widehat{\gamma} \neq 0$ *on* $\overline{Q}$,
- *we have* $\sup_{i \in I} \sum_{j \in I} N_{i,j} < \infty$ *and* $\sup_{j \in I} \sum_{i \in I} N_{i,j} < \infty$, *with*

$$N_{i,j} := \left(\frac{w_i \, |T_j|^{\vartheta}}{w_j \, |T_i|^{\vartheta}} \right)^{\tau} \cdot (1 + \| T_j^{-1} T_i \|)^{\sigma} \cdot \left(\fint_{Q_i} \max_{|\alpha| \leq N} \left| \left[\partial^{\alpha} \widehat{\gamma^{(1)}} \right] (T_j^{-1} (\xi - b_j)) \right| \mathrm{d}\xi \right)^{\tau}.$$

Then there is $\delta_0 > 0$ *such that*

For $0 < \delta \leq \delta_0$, *the family* $\Gamma^{(\delta)} = (\gamma^{[i,k;\delta]})_{i,k}$ *is an a.d. of* $\mathcal{D}(\mathcal{Q}, L^p, \ell_w^q)$

with coefficient space $C_w^{p,q}$.